

Exceptional Indices

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ABSTRACT: Recently a prescription to compute the superconformal index for *all* theories of class \mathcal{S} was proposed. In this paper we discuss some of the physical information which can be extracted from this index. We derive a simple criterion for the given theory of class \mathcal{S} to have a decoupled free component and for it to have enhanced flavor symmetry. Furthermore, we establish a criterion for the “good”, the “bad”, and the “ugly” trichotomy of the theories. After interpreting the prescription to compute the index with non-maximal flavor symmetry as a residue calculus we address the computation of the index of the bad theories. In particular we suggest explicit expressions for the superconformal index of higher rank theories with E_n flavor symmetry, i.e. for the Hilbert series of the multi-instanton moduli space of E_n .

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1. Introduction

A very rich and tractable arena for obtaining exact results for quantum field theories in four dimensions is given by the $\mathcal{N} = 2$ supersymmetric conformal field theories. In recent years a certain set of these theories, obtainable by compactifications of $M5$ branes on Riemann surfaces (called theories of class \mathcal{S} [1, 2]), has received special attention. The six dimensional origin equips these theories with several very interesting properties such as generalized S-duality [1] and the AGT relation [3].

An example of an analytic computation which can be performed for theories of class \mathcal{S} is the evaluation of the superconformal index [4, 5]. The superconformal index for vast majority of these theories is completely fixed by merely assuming the existence of S-duality interconnecting them [6, 7]. The purpose of this paper is to extend the prescription for computing the index to an additional sub-class of theories of class \mathcal{S} not connected to others by S-duality transformations. Some of these theories have a Lagrangian description and thus the computation of the index is straightforward. In other cases these theories will be strongly-coupled SCFTs with no direct way to compute the index. A simplest example of the former is the $\mathcal{N} = 4$ SYM and an example of the latter is rank 2 SCFT with E_6 flavor symmetry.

The prescription of [6] to compute the superconformal index of theories of class \mathcal{S} can be shown [7] to follow directly from S-duality properties of the index of the underlying theories [8]. This prescription translates the data needed to define the compactification of the $(2,0)$ theory on a Riemann surface to the index. However, for some compactifications it gives divergent results. These compactifications, defined by a Riemann surface with punctures, have a common feature that they can not be glued to other theories, and thus S-duality can not be used to fix their index. On the other hand, for some of these compactifications it is widely believed that a well behaved four dimensional theory exists. The basic example again is a torus without punctures which is believed to describe $\mathcal{N} = 4$ SYM.

The divergence of the index calculation can be ascribed to the presence of operators with an unexpected assignment of R-charge. In turns, this signals a breakdown of the assumption that the UV R-charge assignment persists in the IR. Similar phenomena were encountered in three-dimensional $\mathcal{N} = 4$ SCFTs [9], concerning the R-charge assignment of BPS monopole operators. In [10] it was argued that much the same problem could affect four-dimensional theories in class \mathcal{S} , concerning the BPS operators which parameterize the Higgs branch.

We will call the theories for which the index diverges “bad”. In this paper we start by giving a simple and precise criterion when the index computation fails to converge. We further define theories as “ugly” if they contain decoupled free matter: again we give a simple criterion for this to happen. Finally, theories which are not bad nor ugly are called “good” ones.

Further, we reinterpret the prescription to compute the index as an iterative process. Starting with the index of a theory with all maximal punctures, index of theories with reduced flavor symmetry is obtained by computing residues of the former. This type of residue calculus was introduced in [7] and is based on the intuition that the theories with non-maximal punctures live “at infinity” in the Higgs branch of the theories with maximal punctures. The specific direction at infinity is selected by looking at the action of the flavor symmetry.

We observe next that in this procedure, at least in some cases, the bad theories are obtained from ugly ones. The index of the ugly theories is well defined and physically sensible. The contribution to the index of the decoupled free hypermultiplet turns out to be directly responsible for the singularity which appears at the next step of the procedure. We can sharpen our prescription if we distinguish the flavor symmetry rotating the free hypermultiplets from the flavor symmetry rotating the interacting part of the SCFT. This will allow us to compute a well-defined index for some of the bad theories.

In particular we suggest explicit expressions for the index of higher rank theories with E_n flavor symmetry. These theories are bad in our classification but can be obtained from ugly ones. The higher rank E_n theories are of particular interest since their Higgs branch is believed to coincide with the moduli space of multi-instantons of E_n (see e.g. [11, 12]).

This moduli space is not very well studied since the ADHM construction here is lacking. A version of the index on which we will concentrate in this paper, the Hall-Littlewood index in the notations of [6], is equal to the Hilbert series of the Higgs branch for quivers associated to genus zero Riemann surfaces. Thus, we conjecture that the index we compute counts holomorphic functions on the moduli space of multi-instantons. Analogous quantity for a single instanton was computed in [13, 14] and matches the index computation of [15, 6].

This paper is organized as follows. In section 2 we review the prescription of [6] to compute the index of theories of class \mathcal{S} . We define the trichotomy between the good, the bad, and the ugly; and then reformulate the prescription as a residue computation. Then in section 3 we discuss theories for which special care has to be taken in applying the prescription. In particular we give an explicit expression for the index of the rank two SCFT with E_6 flavor symmetry. An appendix contains additional technical details of results and claims presented in the bulk of the paper.

2. Reducing the flavor symmetry “one box at a time”

Let us start by reviewing the general prescription to compute the index of theories of class \mathcal{S} . The superconformal index [4, 5] of an $\mathcal{N} = 2$ SCFT can be thought of as a trace over states of the theory in the radial quantization, i.e. a partition function on $S^3 \times S^1$,

$$\mathcal{I} = \text{Tr}(-1)^F \left(\frac{t}{pq} \right)^r p^{j_{12}} q^{j_{34}} t^R \prod_i a_i^{f_i}. \quad (2.1)$$

We denoted as j_{12} as j_{34} the rotation generators in two orthogonal planes: $j_{12} = j_2 + j_1$ and $j_{34} = j_2 - j_1$ with $j_{1,2}$ being the Cartans of the Lorentz $SU(2)_1 \times SU(2)_2$ isometry of S^3 . r is the $U(1)_r$ generator, and R the $SU(2)_R$ generator of R-symmetries. The a_i are fugacities for the flavor symmetry generators f_i . The states which contribute to the index above satisfy

$$E - 2j_2 - 2R + r = 0. \quad (2.2)$$

We have chosen to compute the index with respect to supercharge $\tilde{\mathcal{Q}}_{1-}$ which has the following charges: $j_1 = 0$, $j_2 = -\frac{1}{2}$, $R = \frac{1}{2}$, and $r = -\frac{1}{2}$. All other choices of supercharges give equivalent results.

The general prescription to compute the index of theories of class \mathcal{S} corresponding to three-punctured spheres with punctures defined by auxiliary Young diagrams Λ_ℓ takes the following form [6]

$$\mathcal{I} = \mathcal{N}_N \prod_{\ell=1}^3 \hat{\mathcal{K}}(\Lambda'_\ell(a_\ell)) \sum_{\lambda} \frac{1}{\psi_{\lambda}(t^{\frac{1-N}{2}}, \dots, t^{\frac{N-1}{2}})} \prod_{\ell=1}^3 \psi_{\lambda}(\Lambda_\ell(a_\ell)). \quad (2.3)$$

The sum over $\lambda \equiv (\lambda_1, \dots, \lambda_{N-1}, 0)$ is a sum over Young diagrams with at most $N - 1$ rows, i.e. over irreducible finite representations of $SU(N)$. The prefactors \hat{K} for the maximal punctures are given by elliptic Gamma functions [6, 7]¹

$$\hat{K}(a_1, \dots, a_N) = \prod_{i \neq j} \prod_{m,n=0}^{\infty} \frac{1 - \frac{pq}{t} p^m q^n a_i / a_j}{1 - p^m q^n t a_j / a_i} \equiv \prod_{i \neq j} \Gamma(t a_i / a_j; p, q) . \quad (2.4)$$

The functions ψ_λ are given by an orthonormal set of eigenfunctions of the elliptic Ruijsenaars-Schneider model [7]. The measure under which these functions are orthogonal is given by

$$\hat{\Delta} = \frac{1}{N!} \prod_{i \neq j} \frac{\Gamma(t a_i / a_j; p, q)}{\Gamma(a_i / a_j; p, q)} . \quad (2.5)$$

In the limit $p = 0$ (or $q = 0$) this reduces to Macdonald measure and the functions ψ_λ become Macdonald polynomials [6]. For the discussion of this paper it will be sufficient to consider only the Hall-Littlewood (HL) limit of the index, $p = q = 0$. The HL index for *linear* quivers is actually equivalent [6] to counting of chiral operators modulo superpotential constraints [14], i.e. to the Hilbert series of the Higgs branch.² In principle, the superconformal index is not sensible to the superpotential constraints. However, in the case of linear quivers (for Lagrangian theories and theories connected to those by dualities) the contributions of the constraints to the Hilbert series precisely match the contributions of the fermions to the HL index. For quivers corresponding to higher genus surfaces this is no more true: in general there will be more fermions than constraints. We refer the reader to [6] for a detailed discussion of this issue and in appendix A we give a couple of simple examples of relations between Hilbert series, HL index, and Higgs branch countings.

For the HL index we can explicitly write down the eigenfunctions for $SU(N)$ theories as,

$$\psi^\lambda(x_1, \dots, x_N | \tau) = \mathcal{N}_\lambda(\tau) \sum_{\sigma \in S_N} x_{\sigma(1)}^{\lambda_1} \dots x_{\sigma(N)}^{\lambda_N} \prod_{i < j} \frac{x_{\sigma(i)} - \tau^2 x_{\sigma(j)}}{x_{\sigma(i)} - x_{\sigma(j)}} , \quad (2.6)$$

which are orthonormal under the measure

$$\Delta_{HL} = \frac{1}{N!} \prod_{i \neq j} \frac{1 - x_i / x_j}{1 - \tau^2 x_i / x_j} . \quad (2.7)$$

To avoid square roots in what follows we define

$$\tau = t^{\frac{1}{2}} . \quad (2.8)$$

¹The abundant relevance of elliptic Gamma functions [16] to the superconformal index computations was observed in [17].

²A possible relation of a similar limit of the $\mathcal{N} = 1$ index with the counting problems discussed in [18, 19] was mentioned in [20].

The normalization $\mathcal{N}_\lambda(\tau)$ is given by

$$\mathcal{N}_{\lambda_1, \dots, \lambda_N}^{-2}(\tau) = \prod_{i=0}^{\infty} \prod_{j=1}^{m(i)} \left(\frac{1 - \tau^{2j}}{1 - \tau^2} \right), \quad (2.9)$$

where $m(i)$ is the number of rows in the Young diagram $\lambda = (\lambda_1, \dots, \lambda_N)$ of length i . For $SU(N)$ groups we take Young diagrams with $\lambda_N = 0$ and the product of x_i in (2.6) is constrained as $\prod_{i=1}^N x_i = 1$. Finally, the over-all normalization constant in (2.3) in the HL limit is given by

$$\mathcal{N}_N = (1 - \tau^2)^{2+N} \prod_{j=2}^N (1 - \tau^{2j}). \quad (2.10)$$

The association of flavor fugacities $\Lambda_\ell(a)$ is illustrated in figure 1. The functions ψ_λ take N

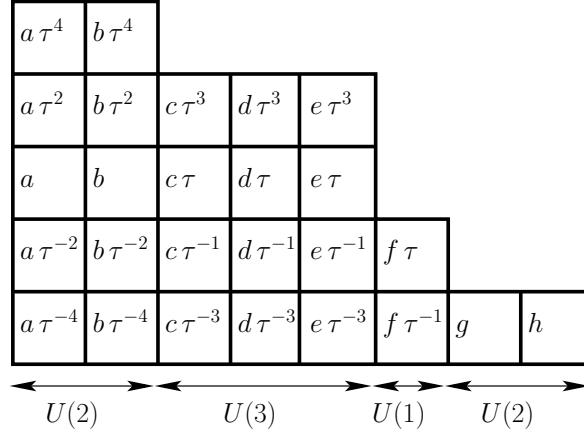


Figure 1: Association of the flavor fugacities for a generic puncture. Punctures are classified by embeddings of $SU(2)$ in $SU(k)$, so they are specified by the decomposition of the fundamental representation of $SU(k)$ into irreps of $SU(2)$, that is, by a partition of k . Graphically we represent the partition by an auxiliary Young diagram Λ with k boxes, read from left to right. In the figure we have the fundamental of $SU(26)$ decomposed as $\mathbf{5} + \mathbf{5} + \mathbf{4} + \mathbf{4} + \mathbf{4} + \mathbf{2} + \mathbf{1} + \mathbf{1}$. The commutant of the embedding gives the residual flavor symmetry, in this case $S(U(3) \times U(2) \times U(2) \times U(1))$, where the $S(\dots)$ constraint amounts to removing the overall $U(1)$. The τ variable is viewed here as an $SU(2)$ fugacity, while the Latin variables are fugacities of the residual flavor symmetry. The $S(\dots)$ constraint implies that the flavor fugacities satisfy $(ab)^5(cde)^4f^2gh = 1$.

arguments. If the flavor symmetry is smaller than $SU(N)$ then the N arguments are read off each box in the auxiliary Young diagram defining the puncture. We associate a flavor fugacity a_i to each column of the auxiliary Young diagram defining the puncture. Furthermore, in each column of height k the boxes are assigned a factor of τ^i ($i = k - 1, k - 3, \dots, 1 - k$).

Similarly, the association of flavor fugacities $\Lambda'_\ell(a)$ is illustrated in figure 2. The difference between the two functions is in the powers of τ . The factors $\hat{\mathcal{K}}(\Lambda'(\mathbf{a}))$ are defined as

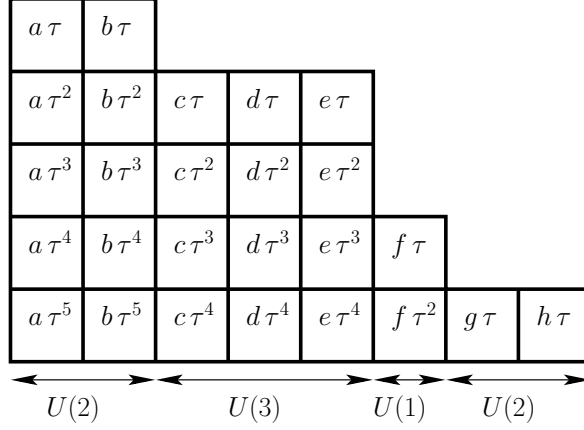


Figure 2: The factors \mathfrak{a}_k^i associated to a generic Young diagram. The upper index is the row index and the lower is the column index. In $\bar{\mathfrak{a}}_k^i$ one takes the inverse of flavor fugacities while τ is treated as real number. As before, the flavor fugacities in this example satisfy $(ab)^5(cde)^4f^2gh = 1$.

$$\hat{\mathcal{K}}(\Lambda'(\mathbf{a})) = \prod_{i=1}^{\text{row}(\Lambda)} \prod_{j,k=1}^{l_i} \frac{1}{1 - \mathfrak{a}_j^i \bar{\mathfrak{a}}_k^i}. \quad (2.11)$$

Here $\text{row}(\Lambda)$ is the number of rows in Λ and l_i is the length of i th row. The coefficients \mathfrak{a}_k^i are assigned as in figure 2.

For HL case the (inverse) structure constants (or the HL quantum dimension) has a simple and explicit form

$$\dim_\tau(\lambda) \equiv \psi_\lambda(\tau^{N-1}, \tau^{N-3}, \dots, \tau^{1-N} | \tau) = \mathcal{N}_\lambda(\tau) \tau^{\sum_{i=1}^{N-1} (2i-N-1)\lambda_i} \prod_{i=1}^N \frac{1 - \tau^{2i}}{1 - \tau^2}. \quad (2.12)$$

Let us finally quote the HL index of the genus \mathfrak{g} theory with s punctures,

$$\mathcal{I}_{\mathfrak{g},s}(\mathbf{a}_I; \tau) = (1 - \tau^2)^{(N-1)(1-\mathfrak{g})+s} \prod_{j=2}^N (1 - \tau^{2j})^{2\mathfrak{g}-2+s} \sum_{\lambda} \frac{\prod_{i=1}^s \hat{\mathcal{K}}(\Lambda'_i(\mathbf{a}_i)) \psi^\lambda(\Lambda_i(\mathbf{a}_i) | \tau)}{[\psi^\lambda(\tau^{N-1}, \tau^{N-3}, \dots, \tau^{1-N} | \tau)]^{2\mathfrak{g}-2+s}}. \quad (2.13)$$

Several examples of using this prescription can be found in [6]. The sum over the representations in (2.13) can be written as a sum over a finite number of geometric progressions. Thus the HL index can be always written as a ratio of two polynomials. However, in practice these polynomials are of high degree and technically hard to evaluate.

2.1 The Good, the Bad, and the Ugly: diagnostics

The prescription of the previous section assumes that the sum over representations in (2.13) converges. This is true in a vast majority of examples where the underlying physical theories are known to exist. However, there are examples where it is widely believed that the $4d$ theory exists but the prescription breaks down. Our purpose will be to understand better these cases and how to adjust the prescription to capture *all* the physically interesting situations.

Let us seek a diagnostic tool to indicate when the sum diverges. It is easy to estimate the leading power of τ in the expansion over the representations. Given a genus \mathfrak{g} A_{N-1} theory with s punctures of arbitrary types and a representation of $SU(N)$ parametrized by $\lambda = (\lambda_1, \dots, \lambda_{N-1}, 0)$, where we always assume $\lambda_i \geq \lambda_{i+1}$, the leading power coming from the HL quantum dimension is

$$\tau^{(2\mathfrak{g}-2+s) \sum_{i=1}^N (N+1-2i)\lambda_i} . \quad (2.14)$$

The leading power from the HL polynomials is given by

$$\tau^{\sum_{i=1}^N (\sum_{\ell=1}^s d_i^{(\ell)}) \lambda_i} , \quad (2.15)$$

where $d_i^{(\ell)}$ are the N different powers of τ for ℓ th puncture which one can read from the auxiliary Young diagram as in figure 1. Moreover these numbers are ordered here

$$d_i^{(\ell)} \leq d_{i+1}^{(\ell)} . \quad (2.16)$$

For example, the puncture depicted in figure 1 gives

$$d = (-4, -4, -3, -3, -3, -2, -2, -1, -1, -1, -1, 0, 0, 0, 0, 1, 1, 1, 1, 2, 2, 3, 3, 3, 4, 4) . \quad (2.17)$$

Thus, we can define a vector fixed by the theory at hand

$$v_i \equiv (2\mathfrak{g} - 2 + s) (N + 1 - 2i) + \sum_{\ell=1}^s d_i^{(\ell)} , \quad (2.18)$$

and the divergences appear if there exists a non-zero vector λ such that

$$v \cdot \lambda \leq 0 . \quad (2.19)$$

Note that if there is one such vector λ automatically there is an infinite number of those by rescaling λ with positive integers. We will call theory satisfying (2.19) a “*bad*” one. As a consistency check note that the contribution of no-puncture (single column) is zero,

$$\delta v_i = N + 1 - 2i + (2(i-1) - N + 1) = 0 . \quad (2.20)$$

Let us give several examples. First, let us consider the extreme case of the torus without punctures for A_{N-1} case. Here the vector v is given by

$$v = 0, \quad (2.21)$$

and the above condition is automatically satisfied for all representations. Here our prescription fails because in the 2d TQFT language of [8] the torus with no punctures is a sum over states in the Hilbert space with a unit weight and such a sum here is infinite. However the index of $\mathcal{N} = 2^*$ SYM corresponding to the torus with one simple puncture is finite since we have the following vector

$$\begin{aligned} i = 1 \dots N-1 : \quad v_i &= N+1-2i+(2(i-1)-N+2) = 1, \\ i = N : \quad v_i &= 1-N. \end{aligned} \quad (2.22)$$

Now since $\lambda_N = 0$ and since the numbers λ_i are non-negative it is obvious that $v \cdot \lambda > 0$. This index is thus converging. Note that for representation $\lambda = (1, 0, 0, \dots)$ we have $v \cdot \lambda = 1$: this will become important momentarily.

Let us consider trinions of $SU(3)$. First, the $SU(3)$ theory corresponding to a sphere with one maximal and two minimal punctures. We get

$$v = (0, 0, 0). \quad (2.23)$$

Here it is obvious that any representation gives $v \cdot \lambda = 0$ and the index is divergent. There is no physical theory corresponding to this surface. Next we consider the free hypermultiplet: sphere with two maximal and one minimal punctures. Here we have,

$$v = (1, 0, -1). \quad (2.24)$$

Since $\lambda_3 = 0$ our criterion gives $v \cdot \lambda > 0$ and the index is convergent. Here also for the choice $\lambda = (1, 0, 0)$ we get $v \cdot \lambda = 1$. Finally the trinion with three maximal punctures gives

$$v = (2, 0, -2). \quad (2.25)$$

Here also obviously $v \cdot \lambda > 0$. Moreover, we do not have any representation giving $v \cdot \lambda = 1$. However, we have $v \cdot \lambda = 2$ for $\lambda = (1, 0, 0)$: this will be also given a meaning shortly.

The next item in the trichotomy we want to define are the “*ugly*” theories. These theories are defined as not bad theories which have a representation for which

$$v \cdot \lambda = 1. \quad (2.26)$$

The physical interpretation is as follows. The HL index gets contributions only from states satisfying [6]

$$E = 2R + r, \quad j_1 = 0, \quad j_2 = r. \quad (2.27)$$

The above condition implies that there is a bosonic state contributing to the index with weight $\tau^{2(E-R)=1}$. Thus this state has to have $j_1 = 0$, $j_2 = r$, $E = 1 - r$, and $R = \frac{1}{2} - r$. Further, unitarity implies³ that the only possible choice is $r = 0$: this is a scalar from the free hyper-multiplet. This means that the theory splits into free components and (possibly) interacting ones. We saw several examples above having this property. The $\mathcal{N} = 2^*$ SYM is $\mathcal{N} = 4$ SYM with decoupled free hyper, and the free hypermultiplet itself. The trinion with maximal punctures however did not have such a term and indeed it does not have free components: it is the interacting E_6 SCFT [21]. Theories which are not bad nor ugly will be called “good” theories.

Finally we have the following additional diagnostic. If the theory is good and if there is a representation such that⁴

$$v \cdot \lambda = 2, \quad (2.28)$$

we expect that the flavor symmetry of the model will be enhanced. The physical motivation for this is that we expect all the operators contributing at τ^2 order to correspond to moment map operators of flavor symmetries. From 2.27 we get that these states have the following charges: $j_i = 0$, $E = 2$, $r = 0$, and $R=1$.

All moment map operators for the naive flavor symmetry are accounted for from the \hat{K} factors in (2.3) and from the sub-leading term in the singlet representation in the sum ($\lambda = 0$); thus any state coming from the sum over non-zero representations enhances the flavor symmetry. If we have an auxiliary Young diagram corresponding to flavor symmetry $\mathcal{G} = S(U(N_1) \times \dots U(N_\ell))$ then the \hat{K} factors contribute the following term at τ^2 order,

$$\sum_{i=1}^{\ell} \sum_{k,j=1}^{N_i} a_j^{(i)} / a_k^{(i)}, \quad (2.29)$$

where $a_j^{(i)}$ are the Cartans of $U(N_i)$. The contributions from the HL quantum dimension and from the HL polynomial corresponding to the singlet representation $\lambda = 0$ are

$$[\dim_{\tau}(0)]^{2-2\mathfrak{g}-s} \rightarrow (1 - \mathfrak{g})(N - 1) - \frac{1}{2}s(N - 1), \quad \psi_0^s \rightarrow \frac{1}{2}s(N - 1). \quad (2.30)$$

³Averaging all the possible $\{Q, Q^\dagger\}$ operators (see [6] for the list of those) in this state and demanding positivity of the result has this implication.

⁴Note that since the ugly theories have decoupled hypermultiplets the flavor symmetry is automatically enhanced since we can rotate the half-hypermultiplets independently of the rest of the theory.

Finally, the over-all normalization factor in (2.13) gives $(N-1)(\mathfrak{g}-1)-s$. Combining all these together we get that at the τ^2 order the contribution to the index from overall factors and from the singlet representation is,

$$\sum_{I=1}^s \left[\sum_{i=1}^{\ell_I} \left(\sum_{k,j=1}^{N_i^{(I)}} a_{(I)j}^{(i)} / a_{(I)k}^{(i)} \right) - 1 \right] \tau^2. \quad (2.31)$$

This is exactly the contribution from the moment map operators of the symmetry \mathcal{G} .

As we saw above we found a non-singlet representation contributing at τ^2 order for the trinion of $SU(3)$ and indeed the flavor symmetry enhances from $SU(3)^3$ to E_6 . On the other hand the trinion of $SU(4)$ with maximal punctures has

$$v = (3, 1, -1, -3), \quad (2.32)$$

and the lowest power one can get here is 3: indeed the $SU(4)^3$ symmetry here does not enhance. For the $SU(4)$ theory with two maximal punctures and one square auxiliary diagram corresponding to $SU(2)$ flavor symmetry we get

$$v = (2, 0, 0, -2). \quad (2.33)$$

Again we get terms of order τ^2 and indeed the symmetry here is enhanced to E_7 .

Let us provide additional examples. We consider an $SU(5)$ theory corresponding to sphere with three punctures: one maximal and two special. The special punctures have two rows with two boxes in the first one and three boxes in the second one. The naive flavor group here is $SU(5) \times SU(2)^2 \times U(1)^2$. Here we are getting

$$v = (2, 0, 0, 0, -2), \quad (2.34)$$

and thus expect the flavor symmetry to be enhanced. Indeed to order τ^2 the index turns out to be

$$\mathcal{I} = 1 + 92\tau^2 + \dots. \quad (2.35)$$

The 92 dimensional adjoint representation is actually an adjoint of $U(1) \times SO(14)$ (see the derivation of this from S-dualities in [22]). In fact there is a whole series of theories corresponding to $SU(2k+1)$ quivers with three punctures: one maximal and two special ones as depicted in figure 3. The theories of this sort were denoted $R_{2,k}$ in [22]. These theories are rank k and the symmetry enhances to $SO(4k+6) \times U(1)$. In the case $k=1$ the $SO(10) \times U(1)$ enhances further to E_6 . Let us compute the v here as another illustration of the procedure. For the two special punctures we have

$$\begin{aligned} k \text{ odd :} \quad & d^{(\ell)} = (1-k, 1-k, 3-k, 3-k, \dots, -1, 0, 1, \dots, k-1, k-1), \\ k \text{ even :} \quad & d^{(\ell)} = (1-k, 1-k, 3-k, 3-k, \dots, 0, 0, 0, \dots, k-1, k-1). \end{aligned} \quad (2.36)$$

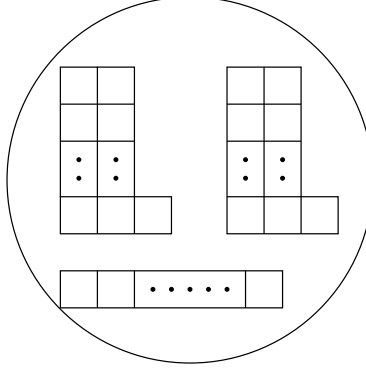


Figure 3: The $SU(2k+1)$ Riemann surface corresponding to the rank k interacting SCFT with flavor symmetry $SO(4k+6) \times U(1)$. The height of the special punctures is k .

For maximal puncture always $d = 0$. Then we get

$$\begin{aligned} k \text{ odd :} \quad & v = (2, 0, 2, 0, 2, 0, \dots, 0, -2, 0, -2), \\ k \text{ even :} \quad & v = (2, 0, 2, 0, 2, 0, \dots, 0, 0, 0, \dots, 0, -2, 0, -2). \end{aligned} \quad (2.37)$$

Another example of $SU(5)$ theory with enhanced symmetry is three punctured sphere with two maximal punctures and an L-shaped one corresponding to symmetry $S(U(1) \times U(2))$. The naive symmetry is $SU(5)^2 \times SU(2) \times U(1)$. However it is enhanced to $SU(10) \times SU(2)$. An example of higher genus theory with enhanced symmetry is genus two surface of $SU(2)$ without punctures,

$$v = (2, -2). \quad (2.38)$$

Here the index is a polynomial

$$\mathcal{I} = 1 + \tau^2 - \tau^4. \quad (2.39)$$

We have a $U(1)$ flavor symmetry giving opposite charges to the two hypermultiplets. This index can not be interpreted as a Higgs branch Hilbert series since it is associated to higher genus surface. Indeed, the τ^4 term comes with negative sign indicating that there are more fermions contributing to the index than there are superpotential constraints. For higher rank genus two surfaces without punctures there is no enhancement of flavor symmetry,

$$v = (2N - 2, 2N - 4, \dots, 2 - 2N). \quad (2.40)$$

Taking A_2 torus with one maximal puncture we have

$$v = (2, 0, -2), \quad (2.41)$$

and thus expect an enhancement of the $SU(3)$ symmetry: indeed it enhances to G_2 . The index actually takes the following form

$$\begin{aligned} \mathcal{I}_{G_2} = & 1 + \chi_{14}^{G_2}(a)\tau^2 + \left(\chi_{Sym^2\mathbf{14}}^{G_2}(a) - \chi_7^{G_2}(a) - 1\right)\tau^4 + \\ & + \left(\chi_{Sym^3\mathbf{14}}^{G_2}(a) - (\chi_7^{G_2}(a) + 1)\chi_{14}^{G_2}(a)\right)\tau^6 + \dots, \end{aligned} \quad (2.42)$$

where

$$\mathbf{14}_{G_2} = \mathbf{3} + \bar{\mathbf{3}} + \mathbf{8}, \quad \mathbf{7}_{G_2} = \mathbf{1} + \mathbf{3} + \bar{\mathbf{3}}. \quad (2.43)$$

Here we start with the A_2 trinion and glue together two legs, i.e. gauge a diagonal $SU(3)$: the commutant subgroup is G_2 . The E_6 adjoint representation decomposes as

$$\mathbf{78} = \mathbf{8} + \mathbf{8}_1 + \mathbf{8}_2 + \mathbf{3} \times \mathbf{3}_1 \times \mathbf{3}_2 + \bar{\mathbf{3}} \times \bar{\mathbf{3}}_1 \times \bar{\mathbf{3}}_2. \quad (2.44)$$

In diagonal $SU(3)'$ we identify $SU(3)_1$ with the conjugate of $SU(3)_2$. Thus, we get a decomposition into G_2 representations,

$$\mathbf{78} = \mathbf{8}' + (\mathbf{8} + \mathbf{3} + \bar{\mathbf{3}}) + \mathbf{8}'(\mathbf{3} + \bar{\mathbf{3}} + \mathbf{1}). \quad (2.45)$$

Note that this is again genus one quiver and thus a-priori we do not have an interpretation of the result as a Hilbert series of the Higgs branch.

2.2 Reducing flavor symmetry and residues: an example

It has been shown in [7] that one can reduce the amount of flavor symmetry of a theory by computing certain residues. We will concentrate in what follows on the HL index, but the residue prescription we will discuss is valid for more refined versions of the index as well. Let us first give a simple example.

We will compute the index of an arbitrary A_{N-1} theory with one L-shaped puncture: the auxiliary Young diagram consists of two rows with one box in the first row and $N-1$ boxes in the second. To do so we will start from the same theory with an L-shaped puncture traded with a maximal one and compute a certain residue. Then we will compare to the prescription of [6] which we reviewed in the beginning of this section.

Let us start with a maximal puncture. The factor $\hat{\mathcal{K}}$ for the maximal puncture has the following form

$$\hat{\mathcal{K}} = \prod_{i,j} \frac{1}{1 - \tau^2 a_i / a_j}. \quad (2.46)$$

This expression has poles whenever

$$a_i = \tau^2 a_j. \quad (2.47)$$

a_1	a_2	a_3	a_4					a_N
-------	-------	-------	-------	--	--	--	--	-------

b_1/τ							
$b_1\tau$	a_2	a_3					a_{N-1}

Figure 4: Getting L-shaped puncture from a maximal one.

Since $\prod_{i=1}^N a_i = 1$ we can also write

$$a_i = \tau \frac{1}{\prod_{j \neq i}^{N-1} a_j^{1/2}}. \quad (2.48)$$

We will treat the fugacities a_1, \dots, a_{N-1} as independent. Let us consider the pole at a_1 and define $b_1 = \prod_{j \neq 1}^{N-1} a_j^{-1/2}$. We put a_1 at the left-most edge of the single row corresponding to the maximal puncture (see figure 4). Then the fugacity assignment in the orthogonal functions, which are HL polynomials in this paper and thus do not have poles themselves, will be

$$\psi^\lambda(a_1, \dots, a_N) \rightarrow \psi^\lambda(\tau b_1, \tau^{-1} b_1, a_2, \dots, a_{N-1}), \quad (2.49)$$

and of course by construction $b_1^2 \prod_{i=2}^{N-1} a_i = 1$. In the prescription of [6] the above assignment corresponds to two row Young diagram with one box in the first row and $N - 1$ boxes in the second and defining a puncture with flavor symmetry $S(U(1) \times U(N - 2))$. Let us now compute the residue of the index at this pole⁵

$$\begin{aligned} \text{Res}_{a_1 \rightarrow \tau b_1} \{\hat{\mathcal{K}}(a_i)\} = & \quad (2.50) \\ \frac{1}{2} \frac{1}{(1 - \tau^2)^2 (1 - \tau^4)} \prod_{i,j=2}^{N-1} \frac{1}{(1 - \tau^2 a_i/a_j)} \prod_{i=2}^{N-1} \frac{1}{(1 - \tau^3 (a_i/b_1)^{\pm 1}) (1 - \tau (b_1/a_i)^{\pm 1})}. \end{aligned}$$

On the other hand the prescription of [6] tells us that the $\hat{\mathcal{K}}$ factor corresponding to the L-shaped puncture is

$$\hat{\mathcal{K}}_L(a_i, b_1) = \frac{1}{(1 - \tau^2)(1 - \tau^4)} \prod_{i,j=2}^{N-1} \frac{1}{(\tau^2 a_i/a_j)} \prod_{i=2}^{N-1} \frac{1}{(1 - \tau^3 (a_i/b_1)^{\pm 1})}. \quad (2.51)$$

The ratio of the two quantities above is simply given by

$$\frac{\hat{\mathcal{K}}_L(a_i, b_1)}{\text{Res}_{a_1 \rightarrow \tau b_1} \{\hat{\mathcal{K}}(a_i)\}} = 2 \mathcal{I}_V \prod_{i=2}^{N-1} (1 - \tau (b_1/a_i)^{\pm 1}). \quad (2.52)$$

⁵We use the short notation implying that when a term with ambiguous signs appears we should consider a product of that term with all possible choices of the sign.

Here, $\mathcal{I}_V = 1 - \tau^2$ is the index of free vector multiplet. Note that the other factor on the right-hand side is an inverse of the index of a free hypermultiplet in representation $(\mathbf{N} - \mathbf{2})_- + \overline{(\mathbf{N} - \mathbf{2})}_+$ of $S(U(1) \times U(N - 2))$. Thus the prescription to obtain the index of a theory with $S(U(1) \times U(N - 2))$ puncture from the index of the theory with the maximal puncture is just to consider the pole described above and multiply the index by the index of a free vector multiplet and divide by the index of an appropriate free hyper-multiplet. This procedure is generalizable to arbitrary punctures by iterating the above and we will explicitly show this in the next section.

In this derivation we assumed that the sum over the representations converges when we consider the special assignment of flavor fugacities and thus the only pole comes from the over-all factors. This fact need not be always true as we saw in the previous subsection. When too many punctures are closed too much this sum does not converge anymore and a good (ugly) theory might become bad. We will discuss this issue in the next section.

2.3 Lifting boxes

Let us now state the general procedure to write the index of a theory with smaller flavor symmetry by a residue computation of a theory with a bigger flavor symmetry.⁶ Given the index, \mathcal{I}_Λ , of a theory with a puncture corresponding to a Young diagram Λ with flavor symmetry \mathcal{G}_Λ we can construct the index, $\mathcal{I}_{\Lambda'}$, of a theory with puncture Λ' differing from Λ by a position of one box and having smaller flavor symmetry,

$$\mathcal{I}_{\Lambda'} = (\ell + 1) \frac{\mathcal{I}_V}{\mathcal{I}_{hyp}(\Lambda, \Lambda')} \operatorname{Res}_{a \rightarrow \tau B} \mathcal{I}_\Lambda. \quad (2.53)$$

The quantity $\mathcal{I}_{hyp}(\Lambda, \Lambda')$ is the index of a free hypermultiplet in a representation we will determine shortly. A reader not interested in the straightforward technical details of the derivation of the above formula can safely skip the rest of this subsection.

Since the residue prescription involves factoring out certain contributions the order in which the procedure is done is essential. We will build the punctures starting from the maximal one by filling up the left-most column first, then going to the next column, and so on. To do so we will raise the right-most box to the desired position. The parameter ℓ in (2.53) is the height of the column to which the box is lifted. From its definition (2.11), the factor of \hat{K} has a pole at the position,

$$a^{\ell+1} = \tau^{\ell+1} B^{\ell+1}. \quad (2.54)$$

⁶The residue prescription here is given for the HL index but it can be generalized to the index with more superconformal fugacities turned on.

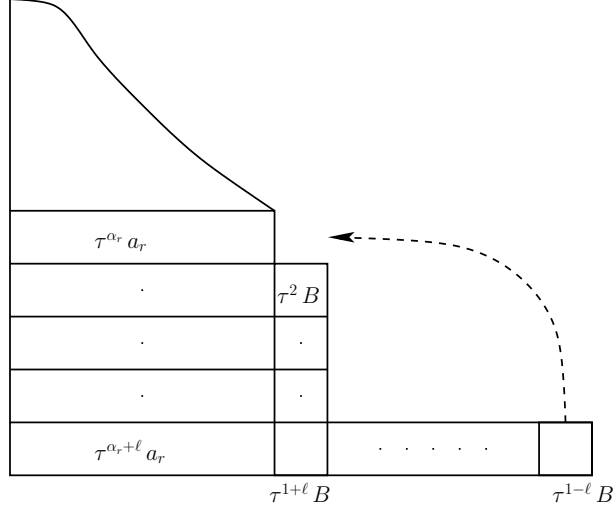


Figure 5: Auxiliary Young diagram. The flavor symmetry is decreased by raising the right-most box to the position indicated by the arrow.

The fugacity a is associated with the column we are lifting the box to. The factor $B^{\ell+1}$ is defined to be the inverse of the product of all the fugacities other than a . If we consider the above mentioned pole then the right-most box, fugacity of which we take to depend on the others, receives the value $\tau^{-\ell} B$, and the fugacity a become τB . Thus in the orthogonal polynomials we obtain

$$\psi^\lambda(\dots, \tau^{\ell-1} a, \tau^{\ell-3} a, \dots, \tau^{1-\ell} a, \frac{B^{\ell+1}}{a^\ell}) \rightarrow \psi^\lambda(\dots, \tau^\ell B, \tau^{\ell-1} B, \dots, \tau^{-\ell} B), \quad (2.55)$$

which exactly corresponds to the Young diagram with the box lifted to the desired position. We will see in what follows that $\mathcal{I}_{hyp}(\Lambda, \Lambda')$ is equal to

$$\mathcal{I}_{hyp}(\Lambda, \Lambda') = \frac{1}{1 - \tau \mathbf{a} b^{-1}} \frac{1}{1 - \tau \mathbf{a}^{-1} b}, \quad b = \tau^{1-\ell} B. \quad (2.56)$$

Here \mathbf{a} are the fugacities of the single row tail of Λ' . Let us derive (2.56). This factor is derived by comparing the residue of $\hat{\mathcal{K}}_\Lambda$ with $\hat{\mathcal{K}}_{\Lambda'}$. Factors of $\hat{\mathcal{K}}$ are obtained by going over each row of the Young diagram and summing over contributions of each ordered pair of boxes, see (2.11). To compare the residue of $\hat{\mathcal{K}}_\Lambda$ with $\hat{\mathcal{K}}_{\Lambda'}$ it is useful to split the auxiliary Young diagram to three regions: (1) the top of the diagram (the curly “hat” in figure 5), (2) the $\ell+1$ lowest rows without the tail (the columns of height one), and the tail (3). The contributions of the rows in the “hat” (1) of figure 5 is the same for both Young diagrams. We only have to compare the $\ell+1$ lowest rows. The contribution of one of the rows in region (2) results in a product of two different types of factors,

$$\frac{1}{1 - \tau^{\alpha_r + 2i-1} a_r / B}, \quad \frac{1}{1 - \tau^{\alpha_r + 2i+1} B / a_r}. \quad (2.57)$$

The latter is the same as the analogous factor in the same row in Λ' but the former actually corresponds to the previous row, $i - 1$, in Λ' . Now the right-most box contracted with the boxes to the left of the column on top of which it is moved gives

$$\frac{1}{1 - \tau^{\alpha_r + 2\ell + 1} a_r / B}, \quad \frac{1}{1 - \tau^{\alpha_r + 1} B / a_r}. \quad (2.58)$$

The former gives the missing factor for the lowest row and the latter gives the missing factor for the row where the extra box is moved to, $(\ell + 1)$ th row. The contractions of the right-most box and the lowest box in the column to which we move the box gives the pole residue of which we compute and

$$\frac{1}{1 - \tau^{2\ell + 2}}. \quad (2.59)$$

This factor gives the correct self contraction of the lowest box in this column in Λ' . The self-contraction of the right-most box gives the factor of

$$\frac{1}{1 - \tau^2}. \quad (2.60)$$

This factor is missing in the diagram Λ' and we have to factor it out. Finally, the contraction of the unit height tail (region (3)) with the right-most box and with the bottom box in the column where we move to yields

$$\frac{1}{1 - \tau^{2+\ell} \mathbf{a} / B} \frac{1}{1 - \tau^{2+\ell} B / \mathbf{a}}, \quad \frac{1}{1 - \tau \mathbf{a} / (B / \tau^{\ell-1})} \frac{1}{1 - \tau (B / \tau^{\ell-1}) / \mathbf{a}}. \quad (2.61)$$

The former factor appears also in Λ' but the latter has to be factored out. To summarize, starting from diagram Λ the residue prescription gives the diagram Λ' and extra factors

$$\frac{1}{1 - \tau^2} \frac{1}{1 - \tau \mathbf{a} / (B / \tau^{\ell-1})} \frac{1}{1 - \tau (B / \tau^{\ell-1}) / \mathbf{a}}. \quad (2.62)$$

The first factor is just the inverse of the index of a free vector and the second is a free hyper in representation $\bar{\mathbf{k}}_{+1} + \mathbf{k}_{-1}$ with the $U(1)$ charge ± 1 coupled to fugacity $B / \tau^{\ell-1}$, and k being the length of the tail of Λ' .

Since the prescription involves taking a residue of $U(1)$ flavor fugacity and multiplying by the index of $U(1)$ vector field it is natural to interpret the procedure as computing the index of the theory corresponding to diagram Λ' by gauging a $U(1)$ flavor symmetry in theory corresponding to diagram Λ . See [7] for a detailed discussion of this intuition.

3. Going beyond the ugly theories

In the previous section we have formulated a prescription to compute the index of a generalized quiver theory by starting from the model with all maximal punctures and performing certain residue computation. The poles we considered appear in pre-factors \hat{K} associated with each puncture. The procedure of taking these residues presumes that the *whole* index has the relevant simple pole. This assumption can fail in two ways,

- The sum over representations has a zero at the relevant value of the flavor fugacity and the pole disappears.
- The sum over representations has a pole at the relevant value of the flavor fugacity and the simple pole becomes higher order pole.

In the following we will see examples of these two situations and discuss the consequences.

3.1 Disappearing poles

An example of the former case is trying to compute the index of a putative theory corresponding to a Riemann surface with one maximal and two minimal punctures. The basic such example is the A_2 case: we will denote such a vertex here as 311. As we saw in section 2.1 this is a bad theory and thus the prescription of [6] fails here. Following the residue prescription of the previous section we obtain the index 311 by starting from the index of 331, the free hyper-multiplet, and computing the residue at the pole $a_1 = \tau B$, where B is a product of fugacities associated to the same puncture as a_1 . The HL index of a free hypermultiplet is given by

$$\mathcal{I}_{331} = \prod_{i,j=1}^3 \frac{1}{1 - \tau a_i b_j c} \frac{1}{1 - \tau \frac{1}{a_i b_j c}}, \quad \prod_{i=1}^3 a_i = \prod_{j=1}^3 b_j = 1. \quad (3.1)$$

However, this index *does not* have the corresponding pole! What happens is that the pole in \hat{K} is canceled out by a zero in the sum over representations. In particular applying the residue prescription we get identically zero implying that there is no theory corresponding to the 311 vertex.

This is consistent with the fact that 311 vertex does not make sense as an independent entity. In particular it is useful to compute it's rank. One can do so by using the Argyres-Seiberg duality [23]. In one duality frame the theory corresponding to four-punctures sphere with two maximal and two minimal punctures is just $N_f = 6$ SYM and thus has rank 2. In the other duality frame, where the two minimal punctures are taken to collide, the same theory can be thought of as the E_6 Minahan-Nemeschanski SCFT [21], which has rank 1, coupled through $SU(3)$ gauge field to 311 vertex. This implies that the naive rank of 311

vertex is -1 .⁷ Thus the vertex 311 only makes sense as a part of bigger theory and does not appear by itself as a sensible quantum field theory.

3.2 Higher order poles

Let us give a couple of examples of cases when the order of the simple pole is increased.

From $\mathcal{N} = 2^*$ SYM to $\mathcal{N} = 4$ SYM

The simplest case of a double pole is $\mathcal{N} = 2^*$ SYM, i.e. the theory corresponding to torus with one minimal puncture. We can try to remove the single puncture of this theory by raising a single box. However, here the sum over representations also has a pole and the corresponding theory is bad. The index of the torus with one minimal puncture is given by (we give the example of A_1 model for simplicity)

$$\mathcal{I}_{\mathcal{N}=2^*} = \frac{1}{1 - \tau a^{\pm 1}} \frac{1 + \tau^2 - \tau^3(a + a^{-1})}{1 - \tau^2 a^{\pm 2}} = \frac{1}{1 - \tau a^{\pm 1}} \mathcal{I}_{\mathcal{N}=4}, \quad (3.2)$$

and indeed this has a second order pole at $a = \tau$. The prescription of [6] to compute the index of the torus thus diverges in this case. Moreover the residue prescription, though giving a finite result, does not make sense physically. The physical reason for this can be traced to the following. The $\mathcal{N} = 2^*$ SYM is actually equivalent to $\mathcal{N} = 4$ SYM with a decoupled hypermultiplet, i.e. it is an ugly theory. Since the free hypermultiplet and the interacting $\mathcal{N} = 4$ SYM are decoupled there is an enhancement of symmetry and it physically actually makes sense to define different flavor fugacity a for the two components,

$$\mathcal{I}_{\mathcal{N}=2^*} = \frac{1}{1 - \tau a_1^{\pm 1}} \frac{1 + \tau^2 - \tau^3(a_2 + a_2^{-1})}{1 - \tau^2 a_2^{\pm 2}}. \quad (3.3)$$

Here, a_1 couples to the $sp(1)$ symmetry rotating the two half-hypers of the free component and a_2 couples to a combination of R-symmetries of the SYM. Thus, after this refinement considering the pole in $a_1 = \tau$ we recover our prescription.⁸ The $\mathcal{N} = 2^*$ is an example of an “ugly” theory since it has decoupled free components and this is exactly the reason why the naive computation of the index is obstructed.

⁷It is interesting to note that our definition of a bad theory here relies on Higgs branch diagnostics. However, in all the cases we considered the bad theories with sphere topology also had a negative “naive” rank [6] (as defined say below equation (1) in [22]) which is a Coulomb branch diagnostics.

⁸The residue at $a_1 = \tau$ gives \mathcal{I}_V^{-1} times the index of $\mathcal{N} = 4$ SYM. Thus, as instructed by (2.53) we strip off the index of the free vector. The Young diagram corresponding to no-puncture does not have a “tail” and thus there is no hypermultiplet to be stripped off. A naive application of (2.53) will also imply that we have to multiply the index by a factor of 2. We do not have to do so here though: as is discussed in [7] such numeric factors are usually associated to “gauging” discrete symmetries, i.e. to having poles also at $a = \exp[\frac{2\pi i k}{\ell+1}]\tau$ for integer values of k ; which we do not have here.

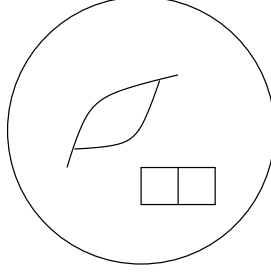


Figure 6: The Riemann surface corresponding to the $\mathcal{N} = 2^*$ theory. The torus without punctures, i.e. the $\mathcal{N} = 4$ SYM is obtained by lifting the right-most box on top of the left-most one.

From A_3 quiver to SYM with $usp(4)$ gauge group

Let us study now another case of a theory with a double pole by considering the example of four punctured sphere of $SU(4)$ type with three square punctures with flavor symmetry $SU(2)$ and one L shaped puncture corresponding to $S(U(1) \times U(2))$ flavor symmetry (let us denote this theory by $222L$). This is an ugly theory of rank two with finite index,

$$v = (2, -1, 1, -2). \quad (3.4)$$

The representation $\lambda = (1, 1, 0, 0)$ gives contribution at order τ and thus we expect to have a free component. We will further try to close the L shaped puncture to a square $SU(2)$ one; the theory with four square-shaped punctures is a “bad” one, index of which naively diverges,

$$v = (2, -2, 2, -2). \quad (3.5)$$

The representations $\lambda = (\ell, \ell, 0, 0)$ all contribute at order τ^0 .

The association of the flavor fugacities to the L -shaped puncture is $(\tau^{-1}b, \tau b, \frac{a}{b}, \frac{1}{ab})$. The limit to 2222 theory, theory with four square punctures, corresponds to taking $a \rightarrow \tau$. Computing the index of $222L$ one can see that the sum over representations has a simple pole in flavor fugacities in the above mentioned limit. Since a simple pole is also present in an over-all factor \hat{K} we have here a double pole. As we mentioned before the HL index is always a ratio of two polynomials. Keeping only the flavor fugacity with respect to which we compute the pole, a , different from one the denominator of the HL index is

$$(a - \tau)^2(a + \tau)(-1 + a\tau)^2(1 + a\tau)(-1 + \tau^2)^{12}(a - \tau^3)^{10}(-1 + a\tau^3)^{10}. \quad (3.6)$$

Here again, the prescription of [6] for the 2222 index diverges since this theory is a bad one, and the residue prescription gives finite but physically non-sensical result.

The 2222 theory is believed [11, 12] to be a $usp(4)$ gauge theory with four hypermultiplets in fundamental representation of the gauge group and one in the antisymmetric representation. Actually computing the index of the $222L$ theory itself one can observe that it is equal

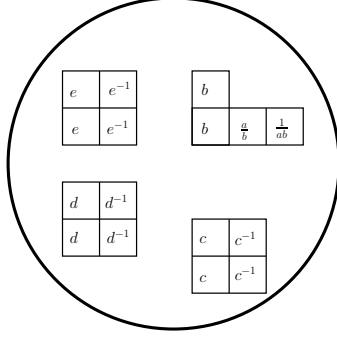


Figure 7: Association of flavor fugacities for the vertex corresponding to the $222L$ theory.

to the index of the $usp(4)$ theory with four fundamental hypermultiplets, one antisymmetric hypermultiplet and a free decoupled hypermultiplet. The four fundamental hypers form a fundamental representation of $SO(8)$ flavor group, which in terms of $SU(2)^4$ maximal sub-group is given by

$$\mathbf{8} = \mathbf{2}_e \times \mathbf{2}_b + \mathbf{2}_c \times \mathbf{2}_d. \quad (3.7)$$

Since the antisymmetric representation is real there is an $sp(1)$ symmetry rotating the two corresponding half-hypermultiplets which we parametrize by fugacity a . Then by explicit computation the index is given by (we bring the technical details in appendix A)

$$\mathcal{I}_{222L}(c, d, e; a, b) = \frac{1}{1 - \tau a^{\pm 1}} \mathcal{I}_{usp(4)+4f+1a}(b, c, d, e; a). \quad (3.8)$$

Here again we can define different a fugacity for the free component and the interacting one resurrecting our prescription. Let us observe that, similarly to what happened in the previous subsection, $usp(4)$ theory has an “accidental” enhancement of flavor symmetry with the addition of $sp(1)$ and thus the $222L$ quiver captures this fact more naturally than the 2222 one.

The pattern emerging from the two examples we discussed is that some of the bad theories which are believed to have good physical description have two useful properties. First, the naive flavor symmetry is enhanced: in the case of $\mathcal{N} = 4$ SYM that was the bigger R-symmetry of the SUSY algebra, and in the $usp(4)$ example of this section the $sp(1)$ symmetry rotating the antisymmetric hypermultiplet. Moreover, these bad theories appear as an interacting component of an ugly theory. Thus if such a situation occurs, the index of the bad theory can be directly inferred from the index of the ugly one. We will use this observation in the next section to compute an index of a bad theory which we did not know a-priori.

The HL index of the $usp(4)$ theory is actually equal [6] to the Hilbert series of the Higgs branch of the same theory [14]. As such, it counts the holomorphic functions on the two-instanton moduli space of $SO(8)$ on $\mathbb{R}^4 = \mathbb{C}^2$.⁹ In this context the decoupled free hypermultiplet in (3.8) is naturally interpreted as the center of mass degree of freedom [24, 14]. Moreover the extra $sp(1)$ symmetry is naturally interpreted as the one rotating the two complex planes of \mathbb{C}^2 [24, 14]. By the same logic HL index of $N_f = 4$ $SU(2)$ theory describes the one-instanton moduli space. In one-instanton case the $sp(1)$ symmetry only affects the center of mass degrees of freedom and thus the centered moduli space did not have this symmetry acting non-trivially on it: for multi-instantons it plays the role for both center of mass and the centered moduli spaces.

It is interesting to note that the coefficient of the leading second order singularity at $a = \tau$ in (3.8) is

$$\frac{(\tau + \tau^3)^2 (1 + 17\tau^2 + 48\tau^4 + 17\tau^6 + \tau^8)^2}{2(-1 + \tau^2)^{22}} = \frac{1}{2} \frac{\tau^2}{(1 - \tau^2)^2} \left(\mathcal{I}_{N_f=4, SU(2)} \right)^2, \quad (3.9)$$

where for simplicity we set all the flavor fugacities to one. Here $\mathcal{I}_{N_f=4, SU(2)}$ is the index of $N_f = 4$ $SU(2)$ SYM. The above result is thus physically naturally interpreted as “going” to the locus in the two-instanton moduli space where the two instantons are far away from each other and the moduli space looks just as a product of moduli spaces of two one-instantons. Moreover, to the lowest orders in τ expansion the index of the $usp(4)$ theory looks like the symmetric product of indices of two $N_f = 4$ $SU(2)$ SCFTs,

$$\begin{aligned} \frac{1}{1 - \tau x^{\pm 1}} \mathcal{I}_{usp(4)+4f+1a}(\tau) = \\ \frac{1}{2} \left[\left(\mathcal{I}_{N_f=4, SU(2)}(\tau) \frac{1}{1 - \tau x^{\pm 1}} \right)^2 + \mathcal{I}_{N_f=4, SU(2)}(\tau^2) \frac{1}{1 - \tau^2 x^{\pm 2}} \right] + O(\tau^4). \end{aligned} \quad (3.10)$$

In particular up to order τ^4 the terms appearing in the index describe generators of the Higgs branch. Thus, the parameters of the two-instanton moduli space are those of the symmetric product of two one-instantons and the details of the geometry, encoded in the Higgs branch constraints, are different between the two.

3.3 The index of rank two E_6 SCFT

It is believed [11, 12] that the A_5 theory corresponding to three punctured sphere with three identical punctures with $SU(3)$ flavor symmetry (two rows with three boxes each) describes an SCFT with E_6 flavor symmetry and rank two. This is a bad theory,

$$v = (2, 0, -2, 2, 0, -2), \quad (3.11)$$

⁹In general it is believed that the Hilbert series of the $usp(2k)$ theory with one antisymmetric hypermultiplet and four fundamental hypermultiplets is related to the moduli space of k instantons.

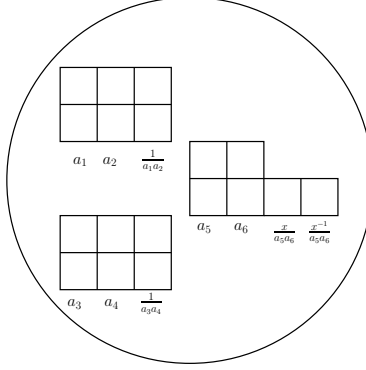


Figure 8: A vertex relevant for the two-instanton moduli space of E_6 .

with representation $\lambda = (\ell, \ell, \ell, 0, 0, 0)$ causing the divergency. Thus, the prescription of [6] here gives a divergent result. One can try to obtain the index of this theory by the residue technology described in this paper but at the final step (the theory depicted in figure 8) a double pole is encountered. However, based on the results of the previous subsection, and in analogy to $\mathcal{N} = 4$ SYM and the rank two $SO(8)$ SCFT, it is tempting to suggest that the theory of figure 8 is *itself* the rank two E_6 SCFT with decoupled hyper-multiplet. The theory is an ugly one,

$$v = (2, 0, -1, 1, 0, -2), \quad (3.12)$$

with representation $\lambda = (1, 1, 1, 0, 0, 0)$ giving contribution at order τ . The flavor symmetry here is $SU(3) \times SU(3) \times S(U(2) \times U(2))$ and we again expect the ugly theory to be equal to the bad one with an addition of a free hypermultiplet. The rank of this theory is two as desired and the Coulomb branch generators have dimensions three and six. Using the general definitions of [6] reviewed here in section 2 the index of this theory is given by

$$\begin{aligned} \mathcal{I} = & \mathcal{N}_6 \mathcal{K}_1(a_1, a_2) \mathcal{K}_1(a_3, a_4) \mathcal{K}_2(a_5, a_6, x) \sum_{\lambda} \frac{\psi_{\lambda}(\tau a_5, \tau^{-1} a_5, \tau a_6, \tau^{-1} a_6, \frac{x}{a_5 a_6}, \frac{x^{-1}}{a_5 a_6} | \tau)}{\psi_{\lambda}(\tau^{-5}, \tau^{-3}, \tau^{-1}, \tau^1, \tau^3, \tau^5 | \tau)} \times \\ & \psi_{\lambda}(\tau a_1, \tau^{-1} a_1, \tau a_2, \tau^{-1} a_2, \tau \frac{1}{a_1 a_2}, \tau^{-1} \frac{1}{a_1 a_2} | \tau) \psi_{\lambda}(\tau a_3, \tau^{-1} a_3, \tau a_4, \tau^{-1} a_4, \tau \frac{1}{a_3 a_4}, \tau^{-1} \frac{1}{a_3 a_4} | \tau). \end{aligned} \quad (3.13)$$

Here $\lambda = (\lambda_1, \dots, \lambda_5, 0)$ and $(b_3 \equiv \frac{1}{b_1 b_2})$

$$\begin{aligned} \mathcal{K}_1(b_1, b_2) &= \prod_{\ell=1}^2 \prod_{i,j=1}^3 \frac{1}{1 - \tau^{2\ell} b_i / b_j}, \\ \mathcal{K}_2(b_1, b_2, x) &= \left[\prod_{\ell=1}^2 \prod_{i,j=1}^2 \frac{1}{1 - \tau^{2\ell} b_i / b_j} \right] \times \\ &\quad \frac{1}{(1 - \tau^2)^2} \frac{1}{1 - \tau^2 x^{\pm 2}} \frac{1}{1 - \tau^3 (b_1^2 b_2 x^{\pm 1})^{\pm 1}} \frac{1}{1 - \tau^3 (b_1 b_2^2 x^{\pm 1})^{\pm 1}}. \end{aligned} \quad (3.14)$$

Expanding this index in power series in τ to lowest orders we obtain

$$\begin{aligned} \mathcal{I} = & \frac{1}{1 - x^{\pm 1} \tau} \left[1 + \left[\chi_{78}^{E_6}(a) + \chi_3^{su(2)}(x) \right] \tau^2 + \chi_{78}^{E_6}(a) \chi_2^{su(2)}(x) \tau^3 + \right. \\ & + \left[\chi_{78}^{E_6}(a) \chi_3^{su(2)}(x) + \chi_{Sym^2 78}^{E_6}(a) + \chi_{Sym^2 \mathbf{3}_{su(2)}}(x) - 1 \right] \tau^4 + \\ & \left. + \left[\chi_{78}^{E_6}(a) \chi_2^{su(2)}(x) \left[\chi_{78}^{E_6}(a) + \chi_3^{su(2)}(x) \right] - \chi_{27}^{E_6}(a) \chi_{\overline{27}}^{E_6}(a) \chi_2^{su(2)}(x) \right] \tau^5 + \dots \right], \end{aligned} \quad (3.15)$$

where we have

$$\begin{aligned} \chi_{27}^{E_6}(a) &= \chi_3^{su(3)}(a_1, a_2) \chi_{\overline{3}}^{su(3)}(a_3, a_4) + \chi_3^{su(3)}(a_3, a_4) \chi_{\overline{3}}^{su(3)}(a_5, a_6) + \chi_3^{su(3)}(a_5, a_6) \chi_{\overline{3}}^{su(3)}(a_1, a_2), \\ \chi_{78}^{E_6}(a) &= \chi_8^{su(3)}(a_1, a_2) + \chi_8^{su(3)}(a_3, a_4) + \chi_8^{su(3)}(a_5, a_6) + \\ &\quad \chi_3^{su(3)}(a_1, a_2) \chi_{\overline{3}}^{su(3)}(a_3, a_4) \chi_3^{su(3)}(a_5, a_6) + \chi_3^{su(3)}(a_1, a_2) \chi_{\overline{3}}^{su(3)}(a_3, a_4) \chi_{\overline{3}}^{su(3)}(a_5, a_6), \\ \chi_{Sym^2 78}^{E_6}(a) &= \frac{1}{2} \left((\chi_{78}^{E_6}(a))^2 + \chi_{78}^{E_6}(a^2) \right). \end{aligned} \quad (3.16)$$

The flavor fugacities a_i form characters of E_6 representations providing a highly non-trivial check of our suggestion. Stripping off the free hypermultiplet we conjecture that we get the index of the rank two E_6 SCFT. The generators of the Higgs branch are visible in the expression at orders $\tau^{2,3}$, and at orders $\tau^{4,5}$ we have constraints appearing. As was the case in the previous example in addition to E_6 flavor symmetry we have more flavor symmetry: here it is $SU(2)$ parametrized by x .

As we mentioned several times before, it was argued in [6] that the HL index for quivers with topology of a sphere is equivalent to the Hilbert series of the Higgs branch. The proof is explicit for theories with Lagrangian description. The argument can be extended also to theories related to Lagrangian by S-dualities. The model in this section can not be connected to Lagrangian theories by S-duality (i.e. it can not be glued to another theory by gauging a maximal puncture due to the simple fact that it lacks a maximal puncture). However, we can still conjecture that also here the HL index is equivalent to the Hilbert series of the Higgs branch. One motivation for this statement is that we obtain the index by a chain of residue

computations each of which can be interpreted as gauging a flavor symmetry. Thus, it would be extremely interesting to study in detail this index because of its possible interpretation as the Hilbert series of the Higgs branch and subsequently as the Hilbert series of the two-instanton moduli space of E_6 .

Let us comment that the above expression (3.15), up to few lowest orders, can be understood as a symmetric product of two rank one E_6 SCFT indices. The Higgs branch and its constraints for this theory were studied in [25] and an elegant expression for the Hilbert series was given recently in [14].¹⁰ This information is also neatly encoded in the index which was computed in [15] and is given by

$$\mathcal{I}_{E_6}^{rank\,1}(a, \tau) = 1 + \chi_{78}^{E_6}(a)\tau^2 + (\chi_{Sym^{27}8}^{E_6}(a) - \chi_{27}^{E_6}(a)\chi_{\overline{27}}^{E_6}(a) + \chi_{78}^{E_6}(a))\tau^4 + \dots \quad (3.17)$$

Taking the symmetric product of this tensored with a free hypermultiplet we obtain

$$\frac{1}{1 - \tau x^{\pm 1}} \mathcal{I}_{E_6}^{rank\,2}(a, \tau) = \frac{1}{2} \left[\left(\mathcal{I}_{E_6}^{rank\,1}(a, \tau) \frac{1}{1 - \tau x^{\pm 1}} \right)^2 + \mathcal{I}_{E_6}^{rank\,1}(a^2, \tau^2) \frac{1}{1 - \tau^2 x^{\pm 2}} \right] + O(\tau^4). \quad (3.18)$$

Also here the generators of the two-instanton case seem to be inherited from the symmetric product of two one-instantons. At higher orders in τ the expressions start to deviate from each other due to the different constraint systems.

Of course this method can be applied to obtain higher rank E_6 theories and also higher rank $E_{7,8}$ models (rank one theories correspond to Minahan-Nemeschansky SCFTs [21, 26] index of which is computed in [27, 6]). The prescription for E_6 would be to consider an A_{3k-1} theory with two rectangular $SU(3)$ punctures and third puncture obtained from $SU(3)$ rectangular one by lowering a box from the top right corner. The rank three case is illustrated in figure 9. These theories have rank k and we suggest that they are the same as the A_{3k-1} theories with three rectangular $SU(3)$ punctures with an addition of free hypermultiplet. We bring in the appendix a first order check of this proposal for the rank three case of E_6 and the rank two case of E_7 . It is also interesting to inquire whether our rank two E_6 index (3.13) has a nice manifestly E_6 covariant closed form on par with the expression for the rank one Hilbert series appearing in [14, 28].

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¹⁰Such expressions are known in mathematical literature, e.g. [13]. We thank Yuji Tachikawa for pointing this out to us.

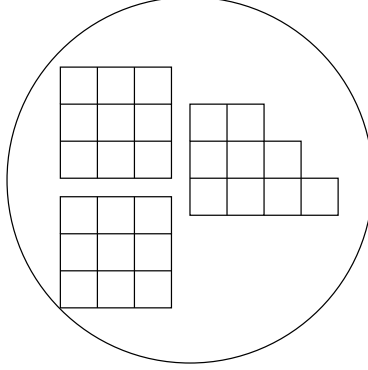


Figure 9: A vertex relevant for the three-instanton moduli space of E_6 .

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A. Technical details

Examples of Hilbert series

Let us give several simple examples of Hilbert series, i.e counting holomorphic functions on complex manifolds. The Hilbert series counts holomorphic functions giving the same weight to functions of same “degree”. Simplest example is functions on the complex plane \mathbb{C} . Such functions are generated by polynomials and the Hilbert series is

$$\mathcal{H} = 1 + t + t^2 + t^3 + \dots = \frac{1}{1-t}. \quad (\text{A.1})$$

Here monomials of degree k , z^k , receive weight t^k . For moduli space of instantons in $\mathbb{R}^4 \sim \mathbb{C}^2$ the relevant series is

$$\mathcal{H} = \frac{1}{1 - x^{\pm 1}t}, \quad (\text{A.2})$$

where we further refined the series by parameter x giving z_1^k weight xt^k and z_2^k weight $x^{-1}t^k$. The parameter x can be viewed as $SU(2)$ fugacity rotating the two copies of the complex plane. This is also of course the HL index of a free hypermultiplet and the Higgs branch Hilbert series. The only field in the hypermultiplet contributing to both objects is a scalar q . The fugacity x couples to the $sp(1)$ symmetry rotating the half-hypers.

Our second example is Hilbert series of two complex planes glued at the origin, $\mathcal{A} = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \mid z_1 z_2 = 0\}$.¹¹ The Hilbert series is thus given by,

$$\mathcal{H} = \frac{1}{1-t_1} + \frac{1}{1-t_2} - 1 = \frac{1-t_1 t_2}{(1-t_1)(1-t_2)}, \quad (\text{A.3})$$

Here t_i couple to the coordinates z_i of the two copies of the complex plane. The constraint $1-t_1 t_2$ implements the fact that the two planes are glued at the origin, $z_1 z_2 = 0$.¹² Taking $t_i \rightarrow x^{\pm 1} t$ we can write the Hilbert series as,

$$\mathcal{H} = \frac{1-t^2}{1-x^{\pm 1} t}. \quad (\text{A.4})$$

Here again x can be thought of as an $SU(2)$ fugacity. The denominator comes from the hypermultiplet and the numerator from the vector multiplet in the HL index language. We have a hypermultiplet with a quadratic constraint.

Note that in both cases, \mathbb{C} and \mathcal{A} , the degree of the singularity at $t = 1$ of the unrefined series, $x = 1$ is equal to one. This degree is interpreted as the complex dimension of the underlying space which in both cases is equal to one. The difference in the topology is encoded in the different numerators in the two cases. Physically the different topology comes from the different constraint system, or in the HL index language simply because of the different matter content.

The index of a $usp(4)$ theory and the $222L$ theory

Let us compute the HL index of the $usp(4)$ gauge theory we encountered in the bulk of the paper. This theory has four hypermultiplet in the fundamental representation **(4)** of the gauge group and a single hypermultiplet in the antisymmetric representation **(5)**. The adjoint representation of $usp(4)$ has dimension ten. The characters of these representations are

$$\begin{aligned} \chi_{\mathbf{4}} &= x_1 + x_1^{-1} + x_2 + x_2^{-1}, & \chi_{\mathbf{5}} &= 1 + x_1 x_2 + \frac{1}{x_1 x_2} + \frac{x_1}{x_2} + \frac{x_2}{x_1}, \\ \chi_{\mathbf{10}} &= 2 + x_1 x_2 + \frac{1}{x_1 x_2} + \frac{x_1}{x_2} + \frac{x_2}{x_1} + x_1^2 + x_2^2 + x_1^{-2} + x_2^{-2}. \end{aligned} \quad (\text{A.5})$$

The theory has $SO(8)$ flavor symmetry rotating the fundamental half-hypers and an $sp(1)$ flavor symmetry rotating the half-hypers of the antisymmetric hypermultiplet. The HL index

¹¹We are grateful to N. Seiberg for pointing out inconsistencies in this subsection in the previous version of the paper.

¹²Topologically equivalent way to view \mathcal{A} is as $\mathbb{C}/\{0\}$, *i.e.* an annulus. In the language of $\mathbb{C}/\{0\}$ the interpretation is as follows. Holomorphic functions on this space are generated by integer, positive and negative, powers of z . Here t_1 couples to z and t_2 to $1/z$. The constraint $1-t_1 t_2$ implements the fact that $(z)(1/z) = 1$.

of this theory is then given by the following matrix integral

$$\mathcal{I}_{usp(4)+4f+1a}(b, c, d, e; a) = \oint \frac{dx_1}{2\pi i x_1} \oint \frac{dx_2}{2\pi i x_2} \Delta_{usp(4)}(\mathbf{x}) PE \left[\tau \chi_8^{SO(8)} \chi_4(\mathbf{x}) + \tau(a + a^{-1}) \chi_5(\mathbf{x}) \right] PE \left[-\tau^2 \chi_{10}(\mathbf{x}) \right], \quad (\text{A.6})$$

where

$$\chi_8^{SO(8)} = (e + e^{-1})(b + b^{-1}) + (c + c^{-1})(d + d^{-1}), \quad (\text{A.7})$$

$$\Delta_{usp(4)}(\mathbf{x}) = \frac{1}{8x_1^2 x_2^2} (x_1 - x_2)^2 (x_1 - x_1^{-1})^2 (x_2 - x_2^{-1})^2 (1 - x_1 x_2)^2,$$

with the latter being the Haar measure of $usp(4)$. Here $PE[\cdot]$ is the plethystic exponent,

$$PE[f(x, y, \dots)] = \exp \left[\sum_{\ell=1}^{\infty} \frac{1}{\ell} f(x^\ell, y^\ell, \dots) \right]. \quad (\text{A.8})$$

On the other hand, following the prescription reviewed in section 2 the index of the 222L theory is given by,

$$\mathcal{I}_{222L}(c, d, e; a, b) = \mathcal{N}_4 \mathcal{K}_1(c) \mathcal{K}_1(d) \mathcal{K}_1(e) \mathcal{K}_2(a, b) \sum_{\lambda} \frac{\psi_{\lambda}(\tau b, \tau^{-1} b, b^{-1} a, b^{-1} a^{-1} | \tau)}{\psi_{\lambda}(\tau^{-3}, \tau^{-1}, \tau, \tau^3 | \tau)} \times \quad (\text{A.9})$$

$$\psi_{\lambda}(\tau c, \tau^{-1} c, \tau c^{-1}, \tau^{-1} c^{-1} | \tau) \psi_{\lambda}(\tau d, \tau^{-1} d, \tau d^{-1}, \tau^{-1} d^{-1} | \tau) \psi_{\lambda}(\tau e, \tau^{-1} e, \tau e^{-1}, \tau^{-1} e^{-1} | \tau).$$

Here we have defined ($b_1 = b$, $b_2 = 1/b$)

$$\mathcal{K}_1(b) = \prod_{\ell=1}^2 \prod_{i,j=1}^2 \frac{1}{1 - \tau^{2\ell} b_i / b_j}, \quad (\text{A.10})$$

$$\mathcal{K}_2(a, b) = \frac{1}{(1 - \tau^2)^3 (1 - \tau^4) (1 - \tau^3 b^{\pm 2} a^{\pm 1}) (1 - \tau^2 a^{\pm 2})}.$$

Comparing the above expressions order by order in τ we conclude that

$$\mathcal{I}_{222L}(c, d, e; a, b) = \frac{1}{1 - \tau a^{\pm 1}} \mathcal{I}_{usp(4)+4f+1a}(b, c, d, e; a). \quad (\text{A.11})$$

Evaluating this index gives,

$$\mathcal{I}_{222L}(1, 1, 1; 1, 1) = \frac{1}{(1 - \tau)^2} [1 + 31\tau^2 + 56\tau^3 + 495\tau^4 + 1468\tau^5 + 6269\tau^6 + \dots]. \quad (\text{A.12})$$

Here at τ^2 order $31 = 28 + 3$ with 28 being the adjoint of $SO(8)$ and 3 adjoint of $SU(2)$. At τ^3 order $56 = 2 \times 28$ with 2 being fundamental of $SU(2)$ and 28 adjoint of $SO(8)$: and so on.

The index of rank three E_6 SCFT

Let us give the expression for the index of rank two E_6 theory. The Riemann surface is depicted in figure 9. The association of flavor fugacities to columns of the auxiliary Young diagrams is as in figure 8 with the only difference being the box we lower: its fugacity is $\frac{x^{-2}}{a_5 a_6}$. Here the index is given by

$$\begin{aligned} \mathcal{I} = & \mathcal{N}_9 \mathcal{K}_1(a_1, a_2) \mathcal{K}_1(a_3, a_4) \mathcal{K}_2(a_5, a_6, x) \times \\ & \sum_{\lambda} \frac{\psi_{\lambda}(\tau^2 a_5, \tau^{-2} a_5, a_5, \tau^2 a_6, \tau^{-2} a_6, a_6, \tau \frac{x}{a_5 a_6}, \tau^{-1} \frac{x}{a_5 a_6}, \frac{x^{-2}}{a_5 a_6} | \tau)}{\psi_{\lambda}(\tau^{-8}, \tau^{-6}, \tau^{-4}, \tau^{-2}, 1, \tau^2, \tau^4, \tau^6, \tau^8 | \tau)} \times \\ & \psi_{\lambda}(\tau^2 a_1, \tau^{-2} a_1, a_1, \tau^2 a_2, \tau^{-2} a_2, a_2, \tau^2 \frac{1}{a_1 a_2}, \tau^{-2} \frac{1}{a_1 a_2}, \frac{1}{a_1 a_2} | \tau) \\ & \psi_{\lambda}(\tau^2 a_3, \tau^{-2} a_3, a_3, \tau^2 a_4, \tau^{-2} a_4, a_4, \tau^2 \frac{1}{a_3 a_4}, \tau^{-2} \frac{1}{a_3 a_4}, \frac{1}{a_3 a_4} | \tau). \end{aligned} \quad (\text{A.13})$$

Here $\lambda = (\lambda_1, \dots, \lambda_8, 0)$ and $(b_3 \equiv \frac{1}{b_1 b_2})$

$$\begin{aligned} \mathcal{K}_1(b_1, b_2) = & \prod_{\ell=1}^3 \prod_{i,j=1}^3 \frac{1}{1 - \tau^{2\ell} b_i / b_j}, \\ \mathcal{K}_2(b_1, b_2, x) = & \left[\prod_{\ell=1}^3 \prod_{i,j=1}^2 \frac{1}{1 - \tau^{2\ell} b_i / b_j} \right] \frac{1}{(1 - \tau^2)} \frac{1}{1 - \tau^3 (b_1^2 b_2 x^{-1})^{\pm 1}} \frac{1}{1 - \tau^3 (b_1 b_2^2 x^{-1})^{\pm 1}} \times \\ & \frac{1}{(1 - \tau^2)(1 - \tau^4)} \frac{1}{1 - \tau^5 (b_1^2 b_2 x^{-1})^{\pm 1}} \frac{1}{1 - \tau^5 (b_1 b_2^2 x^{-1})^{\pm 1}} \times \\ & \frac{1}{1 - \tau^4 (b_1^2 b_2 x^2)^{\pm 1}} \frac{1}{1 - \tau^4 (b_1 b_2^2 x^2)^{\pm 1}} \frac{1}{1 - \tau^3 x^{\pm 3}}. \end{aligned} \quad (\text{A.14})$$

Expanding this index in power series in τ to lowest orders we obtain

$$\mathcal{I} = \frac{1}{1 - x^{\pm 1} \tau} \left[1 + (\chi_{78}^{E_6}(a) + \chi_3^{su(2)}(x)) \tau^2 + \dots \right].$$

Consistently with our claim this index organizes itself in E_6 representations.¹³ To obtain the index of the rank three E_6 SCFT one strips off the free hypermultiplet.

The index of rank two E_7 SCFT

Let us consider the rank two SCFT with E_7 flavor symmetry. The relevant A_7 rank two

¹³The higher order corrections here are technically hard to compute since the complexity of the HL polynomials grows exponentially with the rank of the group. We used the support for Hall-Littlewood polynomials in Sage (<http://www.sagemath.org>) to generate the polynomials.

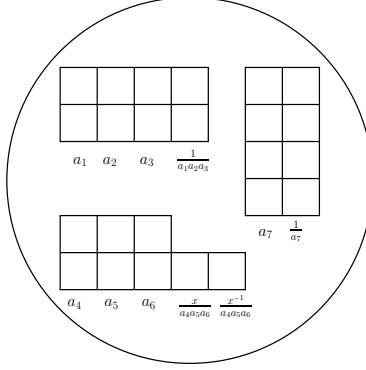


Figure 10: The sphere with three punctures corresponding to the rank two SCFT with E_7 flavor symmetry and decoupled hypermultiplet.

quiver theory is depicted on figure 10. Here the index is given by

$$\begin{aligned} \mathcal{I} = & \mathcal{N}_8 \mathcal{K}_1(a_1, a_2, a_3) \mathcal{K}_2(a_4, a_5, a_6, x) \mathcal{K}_3(a_7) \times \\ & \sum_{\lambda} \frac{\psi_{\lambda}(\tau^3 a_7, \tau^{-3} a_7, \tau a_7, \tau^{-1} a_7, \tau^3 a_7^{-1}, \tau^{-3} a_7^{-1}, \tau a_7^{-1}, \tau^{-1} a_7^{-1} | \tau)}{\psi_{\lambda}(\tau^{-7}, \tau^{-5}, \tau^{-3}, \tau^{-1}, \tau, \tau^3, \tau^5, \tau^7 | \tau)} \times \\ & \psi_{\lambda}(\tau a_1, \tau^{-1} a_1, \tau a_2, \tau^{-1} a_2, \tau a_3, \tau^{-1} a_3, \tau \frac{1}{a_1 a_2 a_3}, \tau^{-1} \frac{1}{a_1 a_2 a_3} | \tau) \\ & \psi_{\lambda}(\tau a_4, \tau^{-1} a_4, \tau a_5, \tau^{-1} a_5, \tau a_6, \tau^{-1} a_6, \frac{x}{a_4 a_5 a_6}, \frac{x^{-1}}{a_4 a_5 a_6} | \tau). \end{aligned} \quad (\text{A.15})$$

Here $\lambda = (\lambda_1, \dots, \lambda_7, 0)$ and $(b_4 \equiv \frac{1}{b_1 b_2 b_3})$

$$\begin{aligned} \mathcal{K}_3(b) = & \frac{1}{(1 - \tau^2)^2 (1 - \tau^4)^2 (1 - \tau^6)^2 (1 - \tau^8)^2} \prod_{\ell=1}^4 \frac{1}{1 - \tau^{2\ell} b^{\pm 2}}, \\ \mathcal{K}_2(b_1, b_2, b_3, x) = & \frac{1}{(1 - \tau^2)^2 (1 - \tau^4)^3} \left[\prod_{\ell=1}^2 \prod_{i,j=1}^3 \frac{1}{1 - \tau^{2\ell} b_i / b_j} \right] \prod_{i=1}^3 \frac{1}{1 - \tau^3 (x^{-1} b_i / b_4)^{\pm 1}} \times \\ & \prod_{i=1}^3 \frac{1}{1 - \tau^3 (x b_i / b_4)^{\pm 1}} \frac{1}{1 - \tau^2 x^{\pm 2}}, \\ \mathcal{K}_1(b_1, b_2, b_3) = & \left[\prod_{\ell=1}^2 \prod_{i,j=1}^3 \frac{1}{1 - \tau^{2\ell} b_i / b_j} \right]. \end{aligned} \quad (\text{A.16})$$

The expansion of the index is given by

$$\mathcal{I} = \frac{1}{1 - x^{\pm 1} \tau} \left[1 + \left(\chi_{133}^{E_7}(a) + \chi_3^{su(2)}(x) \right) \tau^2 + \dots \right]. \quad (\text{A.17})$$

Here the embedding of the flavor fugacities inside E_7 is

$$\mathbf{133}_{E_7} = \mathbf{3}_{su(2)} + \mathbf{2}_{su(2)} (\mathbf{4}_1 \mathbf{4}_2 + \bar{\mathbf{4}}_1 \bar{\mathbf{4}}_2) + \mathbf{15}_1 + \mathbf{15}_2 + \mathbf{6}_1 \mathbf{6}_2, \quad (\text{A.18})$$

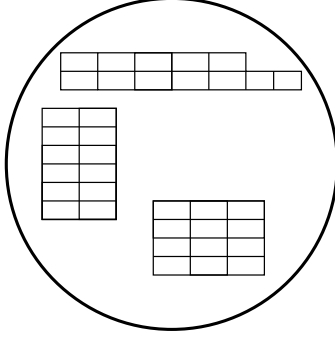


Figure 11: The sphere with three punctures corresponding to the rank two SCFT with E_8 flavor symmetry and decoupled hypermultiplet.

where indices 1 and 2 refer to the two $SU(4)$ groups parametrized by (a_1, a_2, a_3) and by (a_4, a_5, a_6) . The $SU(2)$ is parametrized by a_7 .

Similar expression can be written also for the E_8 higher rank theories. For instance the rank two case is depicted in figure 11. The index can be calculated following our usual prescription.

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